Journal of Global Optimization **16:** 33–41, 2000.

On Generalized Vector Equilibrium Problems

OUAYL CHADLI and HASSAN RIAHI
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. S 15, Marrakech, Morroco.

(Received 16 November 1998; accepted in revised form 20 May 1999)

Abstract. In this paper we prove the existence of solutions of the generalized vector equilibrium problem in the setting of Hausdorff topological vector spaces. As applications, we present some relevant particular cases: a generalized vector variational-like inequality in Hausdorff topological vector spaces, and equilibrium problem in the case of pseudomonotone real functions, and a generalized weak Pareto optima problem.

1. Introduction

Let K be a nonempty set and $f : K \times K \to \mathbb{R}$ be a real function defined on $K \times K$. The equilibrium problem with respect to f and K is defined to be the problem of finding a point $x \in K$ such that $f(x, y) \geqslant 0$ for each $y \in K$. Equilibrium problems are related to numerous important topics of nonlinear analysis in optimization problems, variational inequalities, game theory, complementarity systems, engineering, physics problems, etc. The literature concerning this matter is extensive because of the wide variety and the nonlinearity of problems (see [6, 7] and the references given there). The aim of this paper is to present an overview of a generalized vector equilibrium problem where f is supposed to be a vector valued function defined on a topological vector space. For a treatment of a similar kind of vector equilibrium problems we refer* the reader to [2, 5, 8, 13–18]. The paper is divided into three sections. In Section 2, we give an exposition of a generalized vector equilibrium problem, and present some preliminaries. We introduce a general pseudomonotonicity, such it was proposed by Karamardian (1976) for real functions (see [12]), and general convexity for vector valued functions. In the third section our main existence result of the generalized vector equilibrium problem is stated and proved. The proof is based on the famous Fan lemma. The last section is devoted to some examples of particular case of the main result. In particular, we show a similar result to the one of Ansari [1] on vector variational-like inequality problems. More precisely, we extend the data on the space X, the subset K, the function f, and restrict ourselves to the case of single-valued map. Also, we obtain as a consequence an existence result for equilibrium problem in

* The forthcoming papers [2, 5, 13 and 16] have been added by following the referee's suggestions.

the case of pseudomonotone real functions (see for more general similar results [4] and [7]). In the last remark, we give an application to vector optimization problems and as a particular case we obtain existence of generalized weak Pareto Optima. A careful comparison with the multivalued case in [15] shows that Corollary 4.3 is optimal in a certain sense. More precisely, we cut down the single-valued setting, while generalizing the conditions of compactness on K and continuity on f.

2. Generalized vector equilibrium problems and notations

From now on, let X and Y be two topological vector spaces, K a closed convex subset of X and f a vector single-valued function defined on $K \times K$. Let C be a multivalued map from K into Y. The following assumptions will be needed throughout this paper: for each
(A) $x \in K$, $f(x, x) = 0$;
(B) $x \in K$, $C(x)$ is a convex open cone in Y such that $-cl(C(x)) \cap (C(x) \cup \{0\}) = \{0\}$; for simplicity we say that $C(x)$ is a pointed cone.
Note that (B) implies $0 \notin C(x)$ and $-cl(C(x)) \cap C(x) = \emptyset$ for all $x \in K$. The generalized vector equilibrium problem ((GVEP) for short) to be discussed in this paper is denoted

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \notin C(\bar{x}) \text{ for each } y \in K.$$

Let us now introduce some definitions we need in the next section. For this let us denote by 2^X the set of all nonempty subsets of X, and by $\text{conv}(A)$ the convex hull of a subset A of X. For a multivalued map $F : X \to 2^Y$, clF point out the multivalued map defined, for each $x \in X$, by $clF(x) := cl(F(x))$, the closure of $F(x)$ in Y.

DEFINITION 1. Let X, Y be two vector spaces and K a subset of X. a) a map $f : K \times K \to Y$ is said to be C-pseudomonotone if for each $x, y \in K$ one has

$$f(x, y) \notin C(x) \Longrightarrow f(y, x) \in clC(x). \tag{1}$$

b) A map $g : K \to Y$ is said to be C-convex if for each $x, y \in K$, $\alpha \in [0, 1]$ one has

$$g(\alpha x + (1 - \alpha)y) - [\alpha g(x) + (1 - \alpha)g(y)] \in C(\alpha x + (1 - \alpha)y) \cup \{0\}$$

REMARKS. (1) Two classes of C-pseudomonotone vector valued maps are introduced and analysed in [16], i.e. for each $x, y \in K$

$$f(x, y) \notin C \Longrightarrow f(y, x) \notin -C \tag{2}$$

or

$$f(x, y) \in -clC \Longrightarrow f(y, x) \in clC. \tag{3}$$

These classes are related to each other by (1) implies (2) and (3).
(2) If we specify in the last definition Y to $\mathbb{R}$ and $C(x)$ to $\mathbb{R}^*_-$ for each $x \in K$, we obtain from (1), (2) and ((3) the existing definition of pseudomonotone real functions [12], i.e. $f(x, y) \geqslant 0$ implies $f(y, x) \leqslant 0$ for each $x, y \in K$. Also, in this particular case, the C-convexity is exactly the classical definition of convexity of real functions.

DEFINITION 2. Let X, Y be two topological vector spaces, K a convex subset of X and $F : K \to 2^Y$ a multivalued map.

F is said to be upper hemicontinuous at $x \in K$ if for every $y \in K$ and every neighborhood V of 0 in Y, there exists $\delta \in (0, 1)$ such that $F(tx + (1 - t)y) \subseteq F(x) + V$ when $t \in [0, \delta)$.

If F is a single-valued map we then say that F is hemicontinuous at X.

DEFINITION 3. A multivalued map $F : A \subset X \to 2^X$ is called KKM-map, if for every finite subset $\{x_1, x_2, \dots , x_n\}$ of A,

$$conv\{x_1, x_2, \dots , x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

We also need the following KKM-Fan Lemma [9].

LEMMA 2.1. *Let A be a subset of a Hausdorff topological vector space E and $F : A \to 2^E$ be a KKM-map, such that for each $x \in A$, $F(x)$ is a closed subset of E and for at least one $x_0 \in A$, $F(x_0)$ is compact. then $\bigcap_{x\in A} F(x) \neq \emptyset$.*

3. Existence result

Summarizing the data of the next section we can state the following main result of this paper.

THEOREM 3.1. *Let X be a Hausdorff topological vector space, Y be a topological vector space, K a nonempty closed convex subset of X. Consider $f : K \times K \to Y$ a vector valued function and $C : K \to 2^Y$ a multivalued map such that for each $x \in K$, $C(x)$ is an open convex pointed cone. Suppose moreover that*

(H_0) *f is C-pseudomonotone;*
(H_1) *for each $x \in K$, $\{y \in K : f(x, y) \in clC(y)\}$ is closed and $f(x, .)$ is C-convex;*
(H_2) *clC is upper hemicontinuous on K;*
(H_3) *for each $y \in K$, $f(., y)$ is hemicontinuous;*
(H_4) *there exist a compact subset $B \subset X$ and a vector $y_0 \in K \cap B$ such that*

$f(x, y_0) \in C(x)$ *for each* $x \in K \setminus B$.

Then (GVEP) has a solution.

Proof. To prove the theorem, we first let the multivalued maps F_1 and F_2 defined for each $y \in K$ by

$$F_1(y) := \{x \in K : f(x, y) \notin C(x)\} \text{ and}$$
$$F_2(y) := \{x \in K : f(y, x) \in clC(x)\}.$$

Note that F_1 and F_2 are defined at all points of K, since the assumption (A) implies that $y \in F_1(y) \cap F_2(y)$ for each $y \in K$.

Saying that $(GVEP)$ has a solution means that $\bigcap_{y\in K} F_1(\dot{y}) \neq \emptyset$. This statement will be proved when we show the inclusions

$$\emptyset \neq \bigcap_{y\in K} clF_1(y) \subset \bigcap_{y\in K} F_2(y) \subset \bigcap_{y\in K} F_1(y).$$

Step 1. The inclusion $F_1(y) \subset F_2(y)$ is a direct consequence of pseudomonotonicity of f. From (H_1) the subset $F_2(y)$ is closed and thus $clF_1(y) \subset F_2(y)$. To prove $\bigcap_{y\in k} F_2(y) \subset \bigcap_{y\in K} F_1(y)$, let $\bar{x} \in \bigcap_{y\in K} F_2(y)$, i.e. for each $y \in K$ one has $f(y, \bar{x}) \in clC(\bar{x})$. Let us fix an element y of K and set $y_t = ty + (1-t)\bar{x}$, for $0 < t < 1$, then $f(y_t, \bar{x}) \in clC(\bar{x})$. By C-convexity of $f(y_t, .)$ one has $f(y_t, y_t) - [tf(y_t, y) + (1-t)f(y_t, \bar{x})] \in C(y_t) \cup \{0\}$. Since $C(y_t)$ and $C(\bar{x})$ are nonempty cones, we obtain

$$\begin{aligned} f(y_t, \bar{x}) - f(y_t, y) &\in \frac{1}{t}\left(f(y_t, \bar{x}) + C(y_t) \cup \{0\}\right) \\ &\subset clC(\bar{x}) + clC(y_t). \end{aligned} \tag{4}$$

By upper hemicontinuity of clC we obtain for a neighborhood V of 0 in Y the existence of some $\delta_V \in (0, 1)$ such that, $clC(y_t) \subset clC(\bar{x}) + V$ for every $t \in (0, \delta_V)$.

Combining the two last inclusions and using the hemicontinuity of f we obtain $-f(\bar{x}, y) \in clC(\bar{x}) + V$.

This still true for every neighborhood V in K, thus $-f(\bar{x}, y) \in clC(\bar{x})$. From what has already been noted in (B), since the cone is pointed, we have

$$f(\bar{x}, y) \notin C(\bar{x}),$$

which is $\bar{x} \in F_1(y)$. We conclude that $\bigcap_{y\in K} F_2(y) \subset \bigcap_{y\in K} F_1(y)$.

Step 2. Let us show that $\bigcap_{y\in K} clF_1(y) \neq \emptyset$. The proof is based on the Fan lemma. To verify that clF_1 is a KKM-map, fix $x_1, x_2, \ldots, x_n \in K$, and suppose the existence of $x \in conv(\{x_1, x_2, \ldots, x_n\})$ such that $x \notin \bigcap_{i=1}^n clF_1(x_i)$. Then there exist $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\lambda_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i x_i$, such that

$$f(x, x_i) \in C(x) \text{ for each } i \in \{1, 2, \ldots, n\}.$$

Since $C(x)$ is a convex cone, it follows that $\sum_{i=1}^{n} \lambda_i f(x, x_i) \in C(x)$. Now $f(x, .)$ is C-convex and $f(x, x) = 0$ we than have

$$-\sum_{i=1}^{n} \lambda_i f(x, x_i) = f(x, x) - \sum_{i=1}^{n} \lambda_i f(x, x_i) \in C(x) \cup \{0\}.$$

This contradicts the fact that $C(x)$ is a pointed cone. The only point remaining concerns the compactness assumption in the Fan Lemma. By (H_4) there exist a compact subset B of X and $y_0 \in K \cap B$ such that $f(x, y_0) \in C(x)$ for each $x \in K \setminus B$. This thus get $F_1(y_0) \subset B$, and gives $cl F_1(y_0)$ is compact; which completes the proof. □

REMARKS. (1) Note that if the convex subset K of X is compact then the hypothesis (H_4) may be suppressed, since in this case one has that $F_2(y)$ is compact for each $y \in K$.
(2) The first assertion in condition (H_1) is fulfilled if the multivalued map clC has a closed graph and, for every $x \in K$, $f(x, .)$ is continuous.

Indeed, let us consider a net (y_i) in $F_2(x) := \{y \in K : f(x, y) \in clC(y)\}$ such that (y_i) converges to some y in K. By continuity of $f(x, .)$ we have $f(x, y) = \lim_i f(x, y_i)$. Since the graph of clC is closed, we conclude that $f(x, y) \in clC(y)$; that is $y \in F_2(x)$. Thus $F_2(x)$ is closed. □

4. Some consequences of the main result

In order to clarify the interest of the generalized vector equilibrium problem, we consider some examples of problems for which we can ensure the existence of solutions by relying on.

4.1. VECTOR VARIATIONAL-LIKE INEQUALITY PROBLEM

As a first example of particular case of the generalized vector equilibrium problem, let us consider the following

$$(VVIP) \text{ find } \bar{x} \in K \text{ such that } \langle T(\bar{x}), \eta(y, \bar{x})\rangle \notin C(\bar{x}) \text{ for each } y \in K.$$

Here $T : X \to L(X, Y)$ is a single-valued map from X into the space $L(X, Y)$ of all linear continuous map from X into Y, K is a subset of X $\langle T(x), y\rangle$ is the evaluation of $T(x)$ in y and $\eta : K \times K \to Y$ is a continuous and affine map with $\eta(x, x) = 0$ for all $x \in K$.

DEFINITION 4. $T : X \to L(Y, X)$ is said to be C-pseudomonotone if the function $f(x, y) = \langle T(x), \eta(y, x)\rangle$ is C-pseudomonotone.

Thus we can state the following

COROLLARY 4.1. *Let X be a Hausdorff topological vector space, Y a topological vector space and K a closed convex subset of X. Suppose that*
(i) $T : X \to L(X, Y)$ is a single-valued hemicontinuous and C-pseudomonotone operator;
(ii) $C : K \to 2^Y$ is a multivalued map such that for each $x \in K$, $C(x)$ is an open convex and pointed cone;
(iii) clC has a closed graph;
(iv) there exist a compact subset B of X and $y_0 \in K \cap B$ such that for each $x \in K \setminus B$ one has

$$\langle T(x), \eta(y_0, x)\rangle \in C(x).$$

Then $(VVIP)$ has at least one solution.

Proof. Let us consider $f : K \times K \to X$ defined by $f(x, y) = \langle T(x), \eta(y, x)\rangle$. Condition (H_2) and (H_4) of the main theorem are satisfied. C-convexity of f in (H_1) and continuity of $f(x, .)$ are deduced from continuity and affiness of η. Assumption (H_3) is an immediate consequence of the hemicontinuity of T and the continuity of η. □

REMARKS. i) This corollary is analogous with an existence result of Ansari [1] for multivalued map. More precisely, Ansari consider the restrictive situation where X is a reflexive Banach space, K is a bounded closed convex subset of X and the multivalued map $W(x) = Y \setminus C(x)$ is upper semicontinuous and concave (i.e. for each $x, y \in K$ and $\alpha \in [0, 1]\alpha W(x) + (1-\alpha)W(y) \supseteq W(\alpha x + (1-\alpha)y)$).
ii) If $\eta(x, y) = x - y$ then $(VVIP)$ is reduced to the following vector variational problem (Q) studied by Chen [8]

(Q) find $\bar{x} \in K$ such that $\langle T(\bar{x}), y - \bar{x}\rangle \notin C(\bar{x})$ for each $y \in K$.

4.2. EQUILIBRIUM PROBLEMS

In order to give concrete examples, let us take $Y = \mathbb{R}$ and $C(x) = \mathbb{R}^*_-$, we then obtain the following classical real equilibrium problem

(EP) find $x_0 \in K$ such that $f(x_0, y) \geqslant 0$ for each $y \in K$.

We obtain

COROLLARY 4.2. *Let X be a Hausdorff topological vector space, K a nonempty closed convex subset of X and $f : K \times K \to \mathbb{R}$. Suppose that*
(i) $f(., y)$ is hemicontinuous for each $y \in K$;
(ii) $f(x, .)$ is convex and lower semicontinuous for each $x \in K$;
(iii) f is pseudomonotone;
(iv) there exist $B \subset X$ compact and $y_0 \in K \cap B$ such that $f(x, y_0) < 0$ for each $x \in K \setminus B$.

Then (EP) possesses a solution.

REMARKS. (1) Many authors have contributed to the study of (EP) in different forms. We can see in this subject recent works of Baiocchi- Capelo [3], Blum-Oettli [6], Bianchi-Schaible [4] and Chadli-Chbani- Riahi [7] and the bibliography therein.
(2) By considering the particular case $f(x, y) = \langle T(x), y - x\rangle$ where T, from X into its topological dual X^*, is an operator of monotone type, the problem $(GVEP)$ becomes the following variational inequality problem

$$(VIP) \text{ find } x_0 \in K \text{ such that } \langle T(x_0), y - x_0\rangle \geqslant 0 \text{ for each } y \in K.$$

To this subject we can consult a recent work of Hadjisavvas and Schaible [10].
(3) If in addition, X is a reflexive Banach space or it's a topological dual of some Banach space, the assumption (H_4) can be replaced by the following weak coercive assumption: exists $y_0 \in K$ such that $\limsup f(x, y_0) < 0$ when $x \in K$ and $\|x - y_0\| \to +\infty$.

Indeed, by hypothesis one can find $R > 0$ such that $f(x, y_0) < 0$ for each $x \in K$ with $\|x - y_0\| > R$. Then weak or weak* compactness of $B(y_0, R) = \{y \in X : \|y - y_0\| \leqslant R\}$ implies condition (iv) of Corollary 4.2

4.3. GENERALIZED PARETO OPTIMA

The third example of vector equilibrium problem, we consider, is the following generalized weak Pareto Optima problem

$$(GWPOP) \text{ find } \bar{x} \in K \text{ such that } \varphi(y) - \varphi(\bar{x}) \notin C(\bar{x}) \text{ for each } y \in K.$$

where X is a Hausdorff topological vector space, Y a topological vector space, K a closed convex subset of X, $\varphi : K \to Y$ is a vector valued function and $C : K \to 2^Y$ is a multivalued map such that for each $x \in K$, the open convex cone $C(x)$ is pointed.

By considering $f(x, y) = \varphi(y) - \varphi(x)$, we obtain as an immediate consequence the following corollary.

COROLLARY 4.3. *Suppose that*
(i) φ is continuous, C-convex.
(ii) clC has a closed graph;
(iii) there exist a B compact subset of X and $y_0 \in K \cap B$ such that

$$\varphi(y_0) - \varphi(x) \in C(x) \text{ for each } x \in K \setminus B.$$

Then $(GWPOP)$ has a solution.

REMARKS. (1) in the particular case where $Y = \mathbb{R}^N$ and for each $u \in K$

$$C(u) := C = \{x \in \mathbb{R}^N : x_i < 0,\ i = 1, \dots, N\},$$

we have under assumptions i) and ii) of the precedent corollary the existence of a weak Pareto Optima, i.e. $\varphi(K) \cap (\varphi(u) + C \cup \{0\}) = \{\varphi(u)\}$.
(2) Corollary 4.3 is a generalization, in the case of single-valued functions, of a result [15, Corollary 5-6 p. 59] on existence of solutions of vector optimization problems.

Acknowledgements

This research was partially supported by Action Intégrée Marrakech-Rabat-Montpellier no. 0849/95. The authors wish to express their thanks to anonymous referees for valuable comments and suggestions for improving an earlier draft of the manuscript.

References

1. Ansari, Q.H. (1995), On generalized vector variational-like inequalities, *Ann. Sci. Math. Québec* 19(2): 131–137.
2. Ansari, Q.H., Oetli, W. and Schlager, D. (1997), A generalization of vectorial equilibria, *Math. Meth. Oper. Res.* 46: 147–152.
3. Baiochi, C. and Capelo, A. (1984), Variational and quasivariational inequalities, *Application to Free-Boundary Problems*, J. Wiley & Sons.
4. Bianchi, M. and Schaible, S. (1996), Generalized monotone bifunctions and equilibrium problems, *J. Optim. Theory Appl.* 90: 31–43.
5. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone function, *J. Optim. Theory Appl.* 92: 527–542.
6. Blum, E. and Oetli, W. (1993), From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, 63: 1–23.
7. Chadli, O., Chbani, Z. and Riahi, H. (1999), Equilibrium problems with generalized monotone functions and applications to variational inequalities, Séminaire d'Analyse Convexe, Montpellier, exposé No 20, 1996; *J. Optim. Theory Appl.* (to appear).
8. Chen, G.Y. (1992), Existence of solutions for a vector variational inequality: An extension of Hartman–Stampacchia theorem, *J. Optim. Theory Appl.* 74: 445–456.
9. Fan, K. (1961), A generalization of Tychonoff's fixed point theorem, *Matematische Annalen*, 142: 305–310.
10. Hadjisavvas, N. and Schaible, S. (1996), Quasimonotone variational inequalities in Banach spaces, *J. Optim. Theory Appl.* 90, 95–111.
11. Hlavač̌ek, I., Haslinger, J., Neč̌as, J. and Lavišek, I. (1988), Solution of Variational Inequalities in mechanics, Springer, Berlin.
12. Karamardian, S. and Schaible, S. (1990), Seven kinds of monotone maps, *J. Optim. Theory Appl.* 66: 37–46.
13. Lee, G.M., Kim, D.S. and Lee, B.S. (1996), On noncooperative vector equilibrium, *Indian J. Pure Appl. Math.* 278: 735–739.
14. Lee, G.M., Kim, D.S., Lee, B.S. and Cho, S.J. (1993), Generalized vector variational inequality and fuzzy extension, *Appl. Math. Lett.* 6: 47–51.

15. Luc, D.T. (1989), Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems, 319, Springer Verlag.
16. Oettli, W. (1997), A remark on vector-valued equilibria and generalized monotonicity, *Acta Math. Vietnamica*, 22: 213–221.
17. Parida, J., Shoo, M. and Kumar, A. (1989), A variational-like inequality problem, *Bull. Austral. Math. Soc.* 39: 225–231.
18. Siddiqi, A.H., Ansari, Q.H. and Ahmad, R. (1997), On vector variational-like inequalities, *Indian J. Pure Appl. Math.*, 28(8): 1009–1016.